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Transport in Random Correlated Fields

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A review is given of recent developments in the diffusion properties of particles in the presence of local random fields as well as the conductivity of the analog random resistor network. The effect of long-range ferro- and antiferro-type correlations between the local fields on the diffusion and conductivity properties is considered. A physical realization for such spatial correlations is diffusion on linear polymers in the presence of external uniform bias field. For this case a universal diffusion law was found independent of the fractal dimension of the polymer chain or the Euclidean dimension in which the polymer is embedded. Recent results for diffusion in two dimensions in the presence of a special case of correlated local fields are also reviewed.

KEY WORDS: Correlated fields; transport; fractal; biased diffusion.

1. INTRODUCTION

Diffusion in the presence of random local fields has been attracted much interest in recent years.[1-15] This area is part of the general field of transport in random media. The random field model in one dimension can be defined as follows. A particle at site i has the probability $p_+ = (1 + E_i)/2$ to step to the right and the probability $p_- = (1 - E_i)/2$ to step to the left. The local bias field E_i can obtain the values $-1 \leqslant E \leqslant 1$ from a given distribution $\rho(E_i)$. The simplest case is that E_i is chosen to be $+E$ with probability c and $-E$ with probability $1-c$, i.e., $\rho(E_i) = c\delta(E_i - E) + (1-c)\,\delta(E_i + E)$. The problem of diffusion in the presence of random fields quenched on the lattice, $\mathbf{F}(\mathbf{r})$, can be formulated in the continuum d-dimensional space in the form of a differential equation. Using the

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Fokker–Planck equation, it can be seen that the probability density $P(\mathbf{r}, t)$ satisfies

$$\frac{\partial P}{\partial t} = D\,\nabla^2 P - \nabla \cdot (\mathbf{F} P) \tag{1}$$

In $d = 1$, $\mathrm{F}(x) = \pm E_i$ taken from the above distribution.

Sinai[1] calculated the mean-square displacement $\langle x^2 \rangle$ exactly of a single random walker in one dimension for the case of symmetric probabilities, i.e., $c = 1 - c = 1/2$, and found a logarithmic time dependence,

$$\langle x^2 \rangle \sim \log^4 t \tag{2}$$

The prefactor depends on the field strength E; for details see ref. 5. In fact, the Sinai result is more general and states that asymptotically $(t \to \infty)$ the distribution of displacements x, $P(x, t)$, is a normalized function and scales with $y \equiv x/\langle x \rangle = x/\log^2 t$. Derrida and Pomeau[2] studied the case of a nonsymmetrical distribution of local bias fields E_i and found for the mean-square displacement a power-law time dependence,

$$\langle x^2 \rangle \sim t^{2\nu}; \qquad \nu = \log[c/(1-c)]/\log[E/(1-E)] \tag{3}$$

The form of the probability density $P(x, t)$ for a random walker to be in site x at time t was studied by Nauenberg[3] (heuristically), Kesten[4] (rigorously), and more recently by Bunde *et al.*[5] (numerically). Kesten found analytically for the displacement x a Poisson distribution when the local fields are symmetric and uniformly distributed between $-1 \leqslant E \leqslant 1$, i.e.,

$$P(x, t) \sim \frac{1}{\log^2 t}\, e^{-c|y|} \tag{4}$$

Bunde *et al.* find numerically a different and more localized form for the case of a single value for E. The differences between the theoretical prediction and the numerical data might be due to a difference between averaging over *typical* (most probable) configurations which were taken in the numerical simulations and averages over *all* configurations taken in the analytical approach. Similar differences were also suggested in a different context by Harris and Aharony.[6] Indications of multifractal features in the moments of $P(x, t)$ were observed numerically by Roman *et al.*[7] and were reported also by Stanley.[8]

The problem of first passage times (FPT) and survival probability of particles diffusing in the presence of traps and random fields was studied by Havlin *et al.*[9] They distinguish between *typical* FPTs, which scale according to $\exp(aL^{1/2})$ (L is the system size), and averages over *all* configurations, which scale as $\exp(aL)$. These results were proven rigorously by Noskowicz and Goldhirsch.[10] Based on the above results, Havlin *et al.* find that the survival probability of an independent particle diffusing in the presence of a finite concentration p of traps in a 1D random field system decreases asymptotically (when averaging over all configurations) with time as a power law, $S(t) \sim t^{-\alpha}$. The exponent α was found to depend on the field strength E and on p, $\alpha = 2\log[1/(1-p)]/\log[(1+E)/(1-E)]$. For short times when typical configurations are considered a log-normal distribution of FPTs was obtained,

$$S(t) \sim \exp[-(\log^2 t)] \tag{5}$$

All the above studies assume that the diffusing particles are independent. What happens when interaction between particles, such as a hard-core interaction, is taken into account? This question was addressed in a recent work by Koscielny-Bunde *et al.*[11] They found that the Sinai law, $\langle x^2 \rangle \sim \log^4 t$, for independent particles is still valid, and that hard-core interaction only modifies the prefactor of $\log^4 t$, which tends to zero when the concentration of the hard-core particles approaches unity.

The results described above are for one-dimensional systems. Much less attension was given to higher dimensions. Fisher[12] and Luck[13] show that the probability density for such a random walk in any dimension corresponds to the Green's function of a nonlinear field theory. When applying a renormalization group theory, they find that the upper critical dimension for diffusion in the presence of random fields is $d = 2$ and above two dimensions diffusion is expected to be regular. In two dimensions, $\mathbf{F}(\mathbf{x})$ may be divided into divergence-free and curl-free components. Fisher *et al.*[12] studied the diffusion properties in various cases. For the curl-free case $\nabla \times \mathbf{F} = 0$, they find that $\langle r^2(t) \rangle$ is subdiffusive and has a logarithmic time correction. The curl-free case is more realistic, since it corresponds to a diffusion of a particle in the presence of a scalar potential V, with $\mathbf{F} = \nabla V$, such as diffusion on a rough surface in the presence of a uniform gravitational field. The case of diffusion in two dimensions in the presence of random fields was studied also by Marinari *et al.*[14]

Below two dimensions, the ε expansion yield a power-law anomaly in time and one dimension is the lower critical dimension. On the other hand, numerical results of Pandey[15] for the mean square displacement $\langle r^2(t) \rangle$ in 3-dimensional systems indicate that transport is more enhanced than regular diffusion. For a rigorous upper bound for the mean square displacement in $d \geqslant 2$, $\langle r^2(t) \rangle \leqslant t^{2\nu}$, where ν is the self-avoiding walk exponent $\nu \cong 3/(d+2)$; see Schwatz and Havlin.[16]

Recently[17] a novel and simple random resistor network which is analogous to the diffusion in a random field system was studied. This is reviewed in detail in Section 2. The Sinai random field model assumes that the field at site i is independent of the field at site j, i.e., the local fields are spatially uncorrelated. This model was generalized recently[17] to include short- and long-range correlations between the random bias fields. Both the diffusion problem and its resistor analog were studied in the presence of those correlations and are reviewed in Section 3. Transport in the presence of correlated random fields can be mapped into the problem of diffusion in chainlike fractal systems in the presence of an external uniform bias field.[18] For the latter case a universal diffusion law is found, $\langle r^2 \rangle \sim \log^2 t$, independent of the fractal dimension of the chain or the dimension in which the chain is embedded. This application will be reviewed in Section 4. When the random fields are correlated even via short-range correlations, deviations from Sinai's law occur.[19] There exists a critical moment $q = q_c$ above which the moments of displacements $\langle x^q \rangle$ scale as a power law of time and *not* logarithmically. This is reviewed in Section 5. Finally, diffusion in the special case of the two-dimensional correlated random field model will be presented in Section 6.

2. THE ANALOG RESISTOR SYSTEM

Consider a set of N resistors connected in series, where the resistance R_j of resistor j is related to the resistances of its neighbors by

$$R_{j+1} = (1+\varepsilon)^{\tau_j} R_j \tag{6}$$

Here $\varepsilon > 0$ is arbitrary, and τ_j is chosen randomly to be $+1$ or -1 (see Fig. 1). Since neighboring resistors may only differ by a factor of $(1+\varepsilon)$, this model ensures a smooth spatial variation of the resistance. Using Eq. (6), the resistance of bond l in the network is

$$R_l = R_1 (1+\varepsilon)^{\sum_{j=1}^{l-1} \tau_j} \tag{7}$$

This model represents a random multiplicative system. This is to be compared to the regular random walk model, which is a random additive process. Note that the set of $\{\tau_j\}$ can be viewed as generating a *walk*;

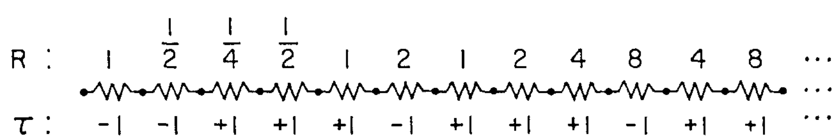

Fig. 1. A realization of the present model with $\varepsilon = 1$.

$\tau_j = +1$ corresponds to a step to the right and $\tau_j = -1$ corresponds to a step to the left.

To see that this resistor network is analogous to diffusion in the presence of a random field, consider the diffusion of a random walker on this system. The probabilities $W_{j,j\pm 1}$ of hopping from site j to its two neighbors are proportional to the inverse of the corresponding resistances between the sites (see Fig. 2),

$$\frac{W_{j,j-1}}{W_{j,j+1}} = (1+\varepsilon)^{\tau_j} \tag{8}$$

From the normalization condition $W_{j,j+1} + W_{j,j-1} = 1$ we obtain $W_{j,j\pm 1} = (1 \pm E)/2$, where $E \equiv \varepsilon/(2+\varepsilon)$, and E plays the role of a *local bias field.* Thus, a random walk on this resistor network system moves according to the rules of a random walker in the presence of a random field. It is therefore expected, due to the relation between conductivity and diffusion, that the laws in both cases will be similar.

The one-dimensional model defined in Eq. (6) includes the essential physics of correlated spatial disorder, but is simple enough to be treated analytically. Note that there exist realistic systems for which this model might be relevant; (i) Measurements of electrical resistance of elongated cylindrical rock samples saturated with salt water, as a function of length. One property of such systems that is already known experimentally[20] and is consistent with our model is the zero-percolation threshold of such

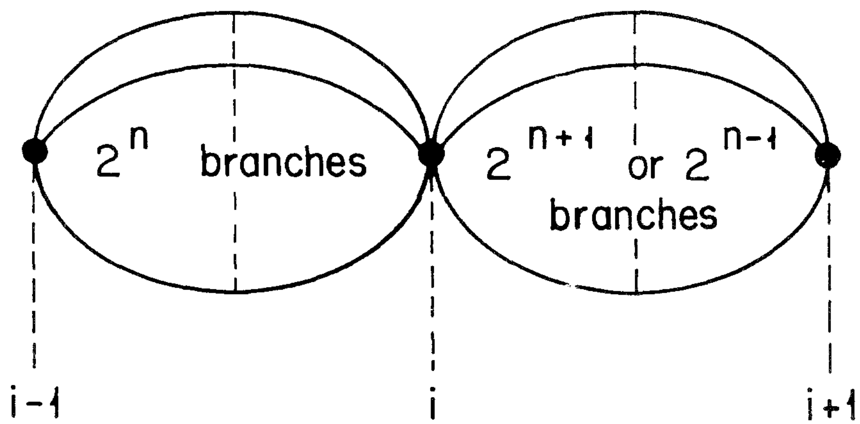

Fig. 2. The connection of the present model to diffusion in the presence of a spatially correlated bias field can be seen by connecting in series 2^{n_j} parallel equivalent branches between each node. The ratio of probabilities $W_{j,j+1}/W_{j,j-1}$ is the ratio of the corresponding number of branches on either side of node j.

systems. (ii) Measurements of resistance to flow in clogged pipes (or "blood vessels"). The process of clogging involves deposition of material on the walls of the pipes, which is faster in and near regions where material is already deposited. The deposition profile may be expected to produce local resistances to flow that are more complex than simple random additive processes.

3. CORRELATED BIAS FIELDS

A quantity of interest is the rms displacement in the N-step walk defined by the set of $\{\tau_j\}$,

$$X(N) \equiv \left\langle \left(\sum_{j=1}^{N} \tau_i \right)^2 \right\rangle^{1/2} \tag{9}$$

In order to calculate $X(N)$, we must defined the distribution of the $\{\tau_j\}$. Consider two cases: (i) $\{\tau_j\}$ uncorrelated and (ii) $\{\tau_j\}$ with short- or long-range spatial correlations. For case (i), where the $\{\tau_j\}$ are uncorrelated, $\langle \tau_i \tau_j \rangle = \delta_{i,j}$ and thus $X(N) = \sqrt{N}$, which is the familiar random walk result. For case (ii), where the $\{\tau_j\}$ have short- or long-range correlations, $X(N)$ depends on the details of the correlations. Assume that our large but finite system is part of a much larger periodic system of size $\Omega \gg N$. We further assume that the distribution is symmetric under reversal of all the τ_j and that the Fourier transforms of the $\{\tau_j\}$, given by

$$\tau_q \equiv \frac{1}{\sqrt{\Omega}} \sum_{l=1}^{\Omega} \tau_l e^{-iql} \tag{10}$$

are correlated through the power-law relation

$$\langle \tau_q \tau_{-q} \rangle \sim \frac{1}{q^\lambda} \qquad \text{for small } q \tag{11}$$

If two neighboring τ_j tend to be of the same sign (which I call the *ferro* case), then $\lambda > 0$; while if two neighboring τ_j tend to be of opposite sign (the *antiferro* case), then $\lambda < 0$. For uncorrelated $\{\tau_j\}$, we have $\lambda = 0$.

It is straightforward to verify that

$$X^2(N) = \frac{1}{\Omega} \sum_q \langle \tau_q \tau_{-q} \rangle \, |f(q, N)|^2 \tag{12}$$

where

$$f(q, N) \equiv \frac{e^{-iq(N+1)} - 1}{e^{-iq} - 1} \tag{13}$$

When $\Omega \to \infty$, we find, on substituting (11) into (12) and converting the sum to an integral, that the dominant contribution scales for large N as

$$X^2(N) \sim N^{1+\lambda}, \qquad 1 \leqslant \lambda \leqslant 1 \tag{14}$$

We will be interested in the result characterizing the total resistance of a chain of length N,

$$R = \sum_{l=1}^{N} R_l = R_1 \sum_{l=1}^{N} (1+\varepsilon)^{\sum_{j=1}^{l-1} \tau_j} \tag{15}$$

To be more specific, an interesting quantity is that of a typical measurement of $R \equiv R_{\text{tot}}(N)$, the total resistance of the N-resistor chain. In a random walk, the mean square displacement, for example, coincides with the most probable value. In the present model, and in random multiplicative processes in general, it is natural to find quantities whose mean and most probable value differ markedly.[21]

If we consider for example the resistance R of the entire chain, we find that its average $\langle R \rangle$ is dominated by improbable configurations of the τ_j (e.g., $\tau_j = 1$ for all j), for which the value of the resistance is large. For case (i), a direct calculation yields $\langle R_l \rangle = \{(1/2)(1+\varepsilon) + (1+\varepsilon)^{-1}\}^{l-1}$, resulting in $\ln\langle R \rangle \sim N$. On the other hand, calculating the average conductivity $\langle 1/R \rangle$ yields[22] $\langle 1/R \rangle \sim N^{-1/2}$. This result is also implied from the dominant contribution of improbable configurations of very small resistances.

The *typical* measured value of R is dominated by its most probable value; the probability of obtaining for R_{tot} a value that differs by an order of magnitude from the most probable value vanishes with N. The *typical* value of a single resistor R_{typ} is represented by the logarithmic average[23]

$$R_{\text{typ}} \sim \exp\langle \ln R \rangle \sim (1+\varepsilon)^{X(N)} \tag{16}$$

The number of such typical resistances (i.e., typical walks) is of the order of N. Hence, from (9) and (16) the logarithmic average $\langle \ln R \rangle$ scales as

$$\langle \ln R \rangle \sim \ln(N \times R_{\text{typ}}) \sim X(N) \ln(1+\varepsilon) + \ln N \tag{17}$$

where the quantity $X(N)$ will depend on the details of the correlations of the τ_j determined above. Note that the quantity $\langle \ln R \rangle$ is sensitive to the details of the ensemble, while the quantity $\langle R \rangle$ is simply dominated by one configuration in the ensemble.

Combining (17) and (14), one finds

$$\langle \ln R \rangle \sim N^{(1+\lambda)/2} + \ln N, \qquad -1 \leqslant \lambda \leqslant 1 \tag{18}$$

The validity of (18) is supported numerically. To this end, we had to generate a set of $\{\tau_j\}$ that have long-range correlations as in (11). Each τ_j configuration is in one-to-one correspondence with an N-step random walk: $X^2(N)$ is the mean square displacement, characterized by the fractal dimension d_w, $X^2(N) \sim N^{2/d_w}$. From (14), $d_w = 2/(1+\lambda)$. Therefore one can generate a distribution with any desired λ $(-1 \leqslant \lambda \leqslant 1)$ by generating the corresponding walk with $1 < d_w < \infty$. We consider the full chain of N sites to consist of strings, each of m sites, where all τ_j in one string have either the same sign (ferro case) or alternating signs (antiferro case). The length m of each string is chosen from the power-law ("Levy flight"; see, e.g., ref. 24) distribution

$$P(m) \sim m^{-\beta} \tag{19}$$

The exponent β determines the correlation parameter λ and therefore d_w. We find that in the ferro case $(\lambda > 0)$

$$d_w = \frac{2}{1+\lambda} = \begin{cases} 1, & \beta \leqslant 2 \\ 2/(4-\beta), & 2 \leqslant \beta \leqslant 3 \\ 2, & \beta \geqslant 3 \end{cases} \tag{20}$$

while in the antiferro case $(\lambda < 0)$

$$d_w = \frac{2}{1+\lambda} = \begin{cases} 2/(\beta-1), & 1 \leqslant \beta \leqslant 2 \\ 2, & \beta \geqslant 2 \end{cases} \tag{21}$$

The antiferro case corresponds to a walk with a singular waiting time distribution, $\Phi(t) \sim t^{-\beta}$; see, e.g., refs. 25 and 26. Equations (20) and (21) are obtained by noting that $\langle \tau_q \tau_{-q} \rangle$ is the Fourier transform of $\langle \tau_0 \tau_l \rangle$ and this is related to $P(m)$ as defined by (19).

The above predictions are supported by computer simulations. Figure 3 is a double logarithmic plot of $\langle \ln R \rangle^2$ and the fluctuation $\langle \ln^2 R \rangle - \langle \ln R \rangle^2$ for the ferro case with $\beta = 1.5$ (corresponding to $\lambda = 1$) as a function of N. Both curves have the same *asymptotic* slope, $(1+\lambda) = 2$, as can be seen by following the procedure used to derive Eq. (18). For $\langle \ln R \rangle^2$, the convergence to the predicted slope is slow due to the $\ln N$ correction in (18), while the fluctuations show the predicted slope already at small values of N.

Figure 4 shows the fluctuation of $\langle \ln R \rangle$ for $\beta = 1.5$, 2.5, and 3.5 in the ferrocase (corresponding to $\lambda = 1$, 1/2, and 0) and for $\beta = 1.5$ and 2.5 in the

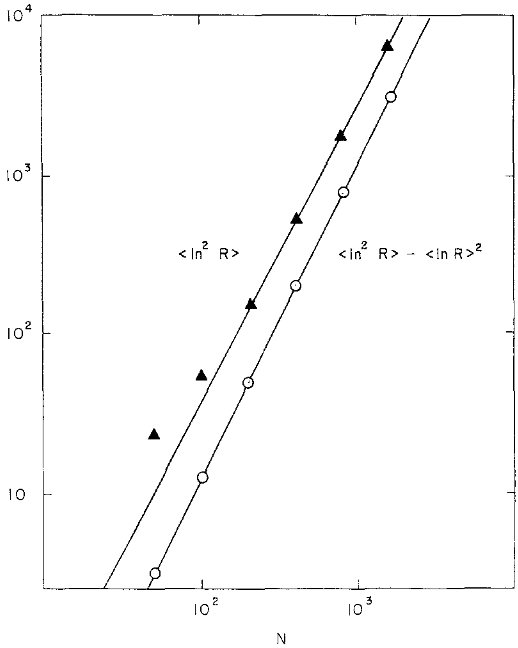

Fig. 3. Plot of $\langle \ln^2 R_{tot} \rangle$ and $\langle \ln^2 R_{tot} \rangle - \langle \ln R_{tot} \rangle^2$ as a function of the size of the system N for $\varepsilon = 0.5$ and $\beta = 1.5$, corresponding to $\lambda = 1$ (ferro case). Note that the best slope to $\langle \ln^2 R_{tot} \rangle$ is slightly lower than the correct value due to the correction term discussed in the text. After ref. 17.

antiferrocase (corresponding to $\lambda = -1/2$ and 0). The numerical results are in excellent agreement with the predictions, Eqs. (18), (20), and (21).

The mean logarithm of the time the random walker takes to travel a distance L along the chain is proportional[25-27] to the fluctuations of the field biased against the walker,

$$\langle \ln t \rangle \sim X(L) \tag{22}$$

Thus, the first passage time in the diffusion problem plays a similar role as the resistance in the resistor problem. Correspondingly, when taking into account correlations between the local bias fields which are determined by the correlations between the τ [see Eq. (11)] one obtains, upon substituting (14) into (22),

$$\langle \ln t \rangle \sim L^{(1+\lambda)/2} \tag{23}$$

The Sinai result is obtained for the average of the displacement for fixed t. If we assume that our result (23) will hold upon fixing t and averaging L,

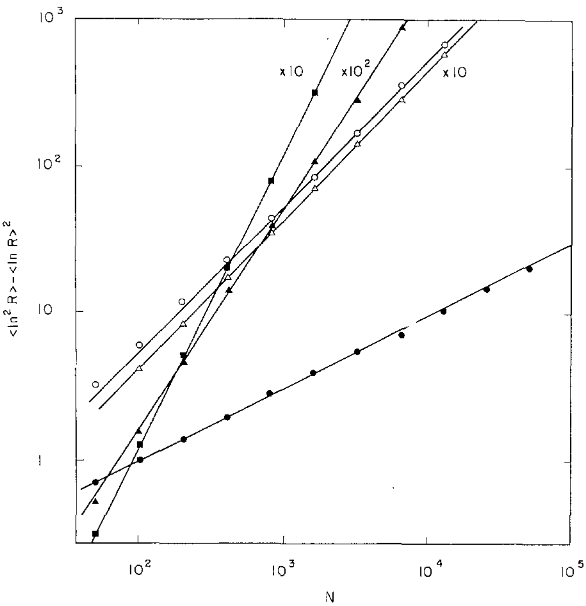

Fig. 4. Double logarithmic plot of $\langle \ln^2 R_{tot} \rangle - \langle \ln R_{tot} \rangle^2$ for $\varepsilon = 0.5$ and various values of β. Ferro case: $\beta = 1.5$ (■), 2.5 (▲), and 3.5 (△). Antiferro case: $\beta = 1.5$ (●) and 2.5 (○). After ref. 17.

we recover the Sinai result in the particular case $\lambda = 0$ (uncorrelated fields). The conditions under which the average can be changed from fixed L to fixed t are discussed in Section 5. Note that the time average $\langle t \rangle$ scales as $\ln\langle t \rangle \sim L$, in analogy with the resistance problem.

4. BIASED DIFFUSION IN CHAINLIKE FRACTAL STRUCTURES

In this section we study random walks on linear fractal structures, such as polymer chains, under the influence of a uniform external field **E**. For a zero field, diffusion is anomalous and the mean square displacement of a walker on the chain *depends* on the fractal dimension d_f of the chain as $\langle r^2(t) \rangle \sim t^{1/d_f}$. When a nonzero external field is applied, it was shown[18] that $\langle r^2(t) \rangle$ is universal and independent of d_f,

$$\langle r^2(t) \rangle \sim \log^2 t \tag{24}$$

To derive Eq. (24), consider a chain of N ordered segments (see Fig. 5). A random walker steps between nearest-neighbor sites on the chain, under the influence of a uniform bias field, that have equal

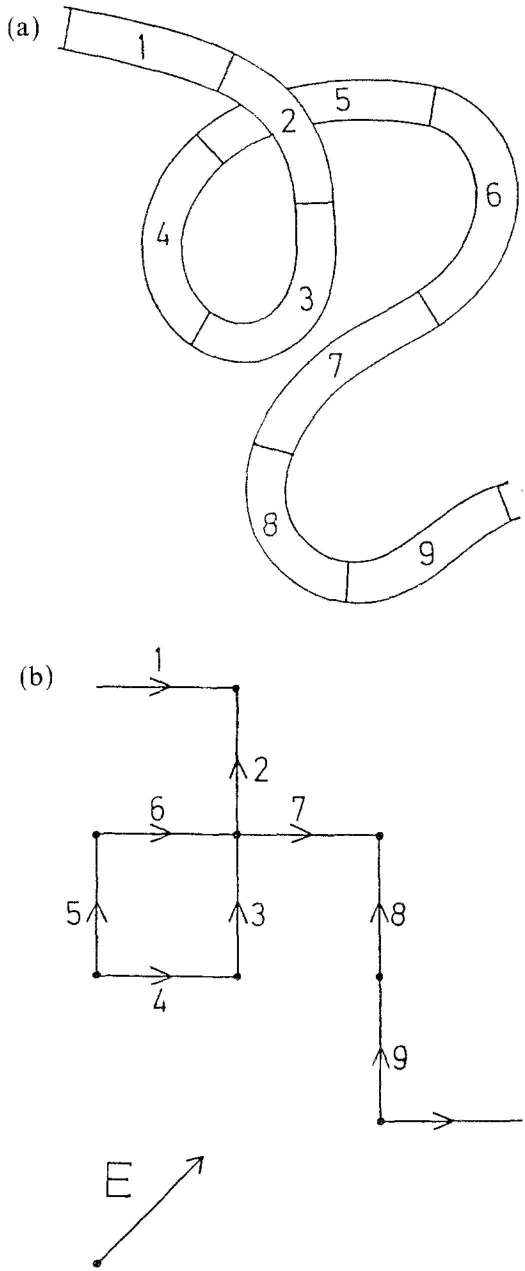

Fig. 5. (a) An illustration of a fractal path consisting of $N = 9$ consecutive ordered segments and (b) the corresponding path on the square lattice. The arrows represent the direction of the local bias fields induced by the external field **E**. After Roman *et al.*[18]

components in the directions of the positive axis. When the particle moves along the path it sees a random field which depends on the direction of the given bond relative to the external field (see Fig. 5). The random field is spatially correlated according to the structure of the chain. Only when the chain is a random walk are the random fields uncorrelated. Thus, assuming that Eq. (23) holds also upon fixing t and averaging over the displacement, it follows that the mean square displacement $\langle l^2 \rangle$ along the path of the chain is given by

$$\langle l^2 \rangle \sim (\log t)^{4/(1+\lambda)} \tag{25}$$

Since $\mathbf{r} = \sum_{u=1}^{l} \mathbf{u}_i$, it follows that $\langle r^2 \rangle = \sum_{i,j=1}^{l} \langle \mathbf{u}_i \cdot \mathbf{u}_j \rangle$. From arguments similar to those leading to Eq. (14), it follows that

$$\langle r^2 \rangle \equiv l^{1+\lambda} \sim l^{2/d_f} \tag{26}$$

The exponent d_f is the fractal dimension of the chain. Substituting Eq. (26) into (25) yields the universal law, Eq. (24).

As an example, let us consider the case where the path is a self-avoiding walk (SAW) in d dimensions. The correlations $\langle \mathbf{u}_i \cdot \mathbf{u}_j \rangle$ have been studied by Domb[28] and the fractal dimension of SAWs is given by the Flory formula, $d_f = (d+2)/3$. Thus, for diffusion on a SAW in $d = 2$ in the presence of a uniform external field we expect

$$\langle l^2 \rangle \sim \log^{8/3} t \tag{27}$$

To test this prediction, $\langle l^2 \rangle$ on self-avoiding walks in $d = 2$ was studied numerically.[18] Long SAWs of about 1000 bonds were generated using the enrichment method.[29] Diffusion on these chains was performed by the exact enumeration method[26] and the second moment $\langle l^2 \rangle$ was calculated. Results averaged over 1000 cofigurations and large fields are plotted as a function of $\log^{8/3} t$ in Fig. 6. The linearity of the curves supports Eq. (27).

5. MOMENTS OF DISPLACEMENT: EFFECTS OF CORRELATIONS

For diffusion in the presence of symmetric random local fields with uncorrelated local fields the probability density $P(x, t)$ is given by Eq. (4). Using this form, it follows that all positive moments behave as

$$\langle |x|^q \rangle \sim \log^{2q} t \tag{28}$$

In this section, I review recent results[19] on the effect of correlations on Eq. (28). The main result is that even for short-range algebraic correlations,

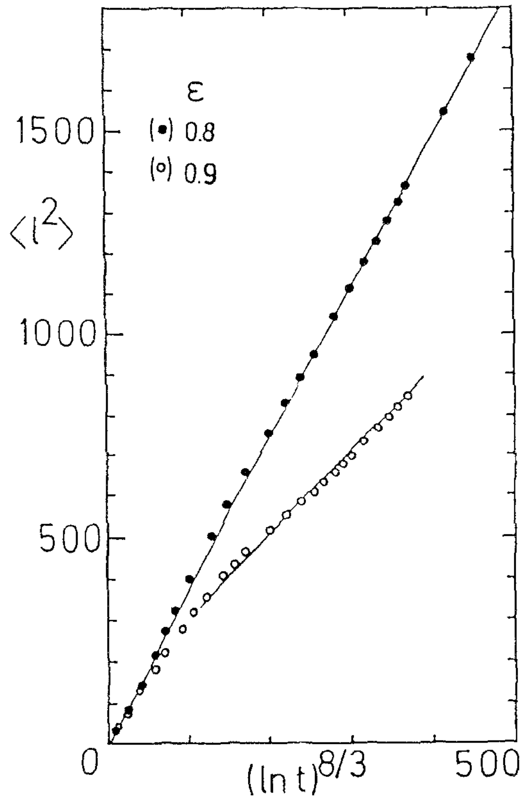

Fig. 6. Plot of the mean square displacement along the path $\langle l^2 \rangle$ as a function of $\log^{8/3} t$ for two values of the bias field **E**. The straight lines support the prediction (27). After Roman *et al.*[18]

Eq. (28) does not hold for all moments. A rigorous proof was given[19] that there exist a critical moment $q = q_c$ above which $\langle |x|^q \rangle$ is a power law of t.

The correlations in the local fields are given by their Fourier components, Eq. (11). The case of uncorrelated random fields is $\lambda = 0$. One might therefore expect that the Sinai generalization (28) will hold for $\lambda = 0$. In fact, it was shown in Section 3 that when averaging $\log t$ for fixed L one obtains $\langle \log t \rangle \sim L^{(1+\lambda)/2}$, a result similar to Sinai's result for $\lambda = 0$. The difference is that Sinai's result corresponds to averaging moments of displacements for fixed t, rather than averaging $\log t$. In the following I prove that Eq. (28) is *not* valid for $\lambda = 0$ when the correlations decay algebraically. Namely, the fact that the correlations are short ranged is not sufficient to ensure Eq. (28) for a general q moment.

Consider the displacement moment $\langle |x|^q \rangle$. The average involved here is a double average. First, the average over different walks is obtained as a functional of the fields, and then one averages over the field configurations. I define the two averages

$$\langle |x|^q \rangle \equiv \overline{\langle |x|^q \rangle_w} \tag{29}$$

where $\langle \cdot \rangle_w$ denotes averaging over the walks and the bar denotes configurational averaging.

Consider now a configuration in which from the initial site of the walker, there is a section of length m taken from the distribution, Eq. (19), to its right. We compare now the actual walks with walks such that the walker stops at the end of the section when arriving there. It is clear that

$$\langle |x|^q \rangle_w \geqslant \langle |x|^q \rangle_{w'(m)} \tag{30}$$

where $w'(m)$ represents the walks that stop at the end of the segment of length m. It is easy to see that

$$\langle |x|^q \rangle_{w'(m)} \sim \begin{cases} t^q, & t \leqslant m \\ m^q, & t > m \end{cases} \tag{31}$$

The upper case follows from the fact that the average displacement in the presence of a uniform field is proportional to t, while the lower case follows from the fact that the walker is trapped at m once it arrives there.

Since $P(m)$ is the probability that the initial string will be of length m, it follows that

$$\langle |x|^q \rangle \geqslant \sum_m P(m) \langle |x|^q \rangle_{w'(m)} \tag{32}$$

Finally, by breaking the above summation into two regimes, we obtain

$$\langle |x|^q \rangle \geqslant \sum_{m<t} P(m) m^q + \sum_{m>t} P(m) t^q \sim t^{q-\beta+1} \qquad \text{for} \quad q-\beta+1>0 \tag{33}$$

Since $\langle |x|^q \rangle$ is bounded from above by t^q, it follows that $\langle |x|^q \rangle$ behaves assymptotically as a power of t, provided that $q > q_c = \beta - 1$. It is seen that even for the case $\lambda = 0$, i.e., $\beta > 3$ as long as β is finite, Eq. (28) is not valid for all moments. Clearly, the field configurations contributing to the right-hand side of Eq. (33) are very rare. If one consider $\langle |x|^q \rangle_w$ for a *typical* configuration, it is expected that

$$\langle |x|^q \rangle_{w(\text{typical})} \sim [\log t]^{2q/(1+\lambda)} \tag{34}$$

Thus, for any finite value of β, one expects a crossover from a logarithmic behavior ($q < q_c = \beta - 1$) to a power-law behavior ($q > q_c = \beta - 1$).

An interesting physical example of this crossover is that of diffusion on a polymer chain in the presence of an external uniform field. The polymer can be described as a d-dimensional self-avoiding walk. For $d < 4$ we find $q_c = 3 - 2/d_f$, where $d_f = 3/(d+2)$ (Flory) is the fractal dimensional of the chain, so that one expects for $q < q_c$ a logarithmic time dependence, and for $q > q_c$ a power-law time dependence. For $d > 4$ it could have been expected that the self-avoiding walk is equivalent to a random walk and therefore Eq. (28) would apply for all q. It can be shown, however, that due to the corrections to a random-walk behavior the mean square end-to-end distance of a self-avoiding walks in $d > 4$ is

$$\langle R^2 \rangle \sim N(1 + A/N^{\alpha(d)}) \tag{35}$$

We find that the β corresponding to the self-avoiding walk is *finite* and related to $\alpha(d)$ via

$$\alpha(d) = \beta - 3 \tag{36}$$

For example, for $d = 5$, $\alpha(5) = 1/2$; thus, $\beta = 7/2$ and $q_c = 5/2$.

6. DIFFUSION IN THE PRESENCE OF RANDOM FIELDS IN $d = 2$: A SPECIAL CASE

In a recent work Blumberg-Selinger *et al.*[30] studied a special case of diffusion in a two-dimensional random field system. In this model the positive x axis represents random potential $V(x, 0)$ with a random gradient ± 1. Similarly, for the positive y axis, one has $V(0, y)$. The potential at a general site $x, y > 0$ is defined as

$$V(x, y) = V(x, 0) + V(0, y) \tag{37}$$

This potential is strongly correlated, and the field $\mathbf{F} = \nabla V(x, y)$. It was found rigorously[30] that for this model Eq. (4) can be decoupled into two independent one-dimensional equations with uncorrelated random fields. Thus, the mean square displacement follows,

$$\langle r^2(t) \rangle \sim \log^4 t \tag{38}$$

similar to the one-dimensional Sinai result, Eq. (2).

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